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COMMENT

Pairs of analytical eigenfunctions for the $x^2+\lambda x^2/(1+gx^2)$ interaction

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Abstract. A general method is discussed for deriving pairs of exact analytical eigenfunctions of the $x^2+\lambda x^2/(1+gx^2)$ interaction, recently discussed by Blecher and Leach and commented on by Gallas. It will be shown that those pairs always consist of an odd- and an even-parity solution. In each case λ and g are related by the conditions $\lambda<0$, $g>0$ and $\lambda=\lambda(g)$. As a result of our investigations we conjecture the possible existence of an infinite number of solution pairs having λ and g connected by several different relations.

1. Introduction

The eigenvalue problem

$$y''+[E-x^2-\lambda x^2/(1+gx^2)]y=0 \qquad g>0 \tag{1.1}$$

has been the subject of several numerical as well as analytical studies in recent years. Specific references to numerical approaches can be found in Fack and Vanden Berghe (1985). A set of exact solutions of (1.1) has been constructed by Flessas (1981, 1982), Varma (1981), Lai and Lin (1982) and Whitehead *et al* (1982). The existence of such exact solutions is related to the conditions $\lambda<0$, $g>0$ and $\lambda=\lambda(g)$, $E=E(g)$. Since those exact solutions require a definite relationship between λ and g, for a given permissible potential one only finds in general just one eigenvalue. Blecher and Leach (1987) argue that for suitable choices of λ and g exact analytical eigensolutions for (1.1) of the form

$$y(x)=\sum_{i=0}^{n} a_i x^{2i+\delta}(1+gx^2)\exp(-x^2/2) \tag{1.2}$$

can be constructed, whereby δ is zero or one depending upon whether one is looking for even or odd wavefunctions. Moreover, they mention the possibility of deriving more than one exact eigenvalue per potential. They show that if one wishes to obtain the same λ for a pair of solutions of the type (1.2) labelled by n_1, δ_1, n_2 and δ_2, the corresponding energies E_1 and E_2 must be related by

$$E_2=E_1+4(n_2-n_1)+2(\delta_2-\delta_1). \tag{1.3}$$

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In a recent paper Gallas (1988) reinvestigates the subject and shows that there exist pairs of solutions of the type

$$y_1 = x(1+gx^2)\exp(-x^2/2) \tag{1.4}$$

$$y_2 = \sum_{i=0}^{n} a_i x^{2i}(1+gx^2)\exp(-x^2/2) \tag{1.5}$$

with $n \in \{1, 2, 3, 4, 5\}$, provided λ and g are connected by the relation

$$\lambda = -(6g^2+4g). \tag{1.6}$$

In these five cases the acceptable value of g is the unique positive root of an nth degree polynomial. Moreover the values of g for the five values of n considered are curiously spaced by an almost constant difference. Gallas conjectures the possible existence of an infinite number of solution pairs of the type (1.4), (1.5) obeying (1.6). In the present comment we shall give a proof for this conjecture. As a byproduct, new pairs of solutions, connected to other relations between λ and g, are reported.

2. General theory

It is important to examine whether two arbitrary solutions of the type (1.2) can form for the same interaction a pair of solutions of (1.1). A general investigation which we shall not develop here in detail shows that this is only possible when an even- and an odd-parity solution are combined. Let us therefore consider the following general eigensolutions:

$$y_1 = \sum_{i=0}^{n_1} a_i x^{2i+1}(1+gx^2)\exp(-x^2/2) \tag{2.1}$$

$$y_2 = \sum_{i=0}^{n_2} b_i x^{2i}(1+gx^2)\exp(-x^2/2). \tag{2.2}$$

Substituting (2.1) into (1.1) and multiplying by $(1+gx^2)$, one obtains from the coefficients of the several powers of x a system of equations of the type

$$-a_i(4ig+7g-\varepsilon_1 g+\lambda)+a_{i+1}[(2i+5)(2i+4)g-4i-7+\varepsilon_1]+a_{i+2}(2i+5)(2i+4)=0 \tag{2.3}$$

which are valid for all $i \in [-1, n_1]$, provided that $a_1 = a_{n_1+1} = a_{n_2+1} = 0$. The last equation, i.e. that for $i = n_1$, related to the highest power of x, gives rise to the relation

$$\varepsilon_1 = 4n_1 + 7 + \lambda/g \tag{2.4}$$

which shows the dependence of the ε_1 eigenvalue on the ratio λ/g. The substitution of (2.4) into the set of equations (2.3) for $i \in [-1, n_1 - 1]$ results in a set of n_1 homogeneous equations in the (n_1+1) coefficients a_i $(i = 0, \ldots, n_1)$. One of these coefficients can be chosen arbitrarily, a fact which is related to the normalisation condition imposed upon eigensolutions of Schrödinger equations. Choosing $a_0 = 1$, the resulting set of n_1 homogeneous relations in $a_1, a_2, \ldots, a_{n_1}$ has a non-trivial solution if and only if the determinant of the coefficient matrix is zero. This determinant condition imposes a relation between λ and g. Defining

$$u = \lambda/2g \tag{2.5}$$

this condition is as follows.

(i) For $n_1=0$:

$$u=-(3g+2) \tag{2.6}$$

which is the relation (1.6), already mentioned by Gallas.

(ii) For $n_1=1$:

$$u^2+(13g+6)u+(30g^2+40g+8)=0 \tag{2.7}$$

resulting in

$$u=\tfrac{1}{2}[-13g-6\pm(49g^2-4g+4)^{1/2}] \tag{2.8}$$

which shows that for each positive value of g two negative values of u (or λ) are obtained.

(iii) For $n_1=2$:

$$u^3+2(17g+6)u^2+(303g^2+276g+44)u+(630g^3+1260g^2+504g+48)=0. \tag{2.9}$$

For positive values of g the discriminant of this cubic equation is strictly negative, indicating the presence of three different negative values of u (or λ).

These three results suggest the possible existence for every value of n_1 of n_1+1 negative values of u (or λ) when g takes on positive values.

Substituting (2.2) into (1.1) and multiplying by $(1+gx^2)$, the separation of coefficients of different powers of x results in a system of equations of the type

$$-b_i(4ig+5g-\varepsilon_2 g+\lambda)+b_{i+1}((2i+3)(2i+4)g-4i-5+\varepsilon_2)+b_{i+2}(2i+3)(2i+4)=0 \tag{2.10}$$

which is valid for all $i\in[-1, n_2]$, provided $b_1=b_{n_2+1}=b_{n_2+2}=0$. For $i=n_2$ one obtains, taking into account (2.5),

$$\varepsilon_2=4n_2+5+\lambda/g=4n_2+5+2u=\varepsilon_1+4(n_2-n_1)-2 \tag{2.11}$$

where we have made use of (2.4). The relation confirms (1.3) of Blecher and Leach (1987).

Substituting (2.11) into (2.10) for all values of i belonging to the interval $[-1, n_2-1]$ and choosing $b_0=1$, one obtains again a set of n_2 homogeneous equations in $b_1,\ldots,b_{n_2}$ for which only non-trivial solutions exist if the following determinant condition holds:

$$\begin{vmatrix} g+2(n_2+1)+u & 1 & 0 & 0\ldots \\ 2gn_2 & 6g+2n_2+u & 6 & 0\ldots \\ 0 & 2g(n_2-1) & 15g+2(n_2-1)+u & 15\ldots \\ & & \vdots & \\ & \ldots 2g(n_2-i) & (2i+3)(i+2)g+2(n_2-i)+u & (i+2)(2i+3)\ldots \\ & \ldots 4g & (2n_2-1)n_2 g+4+u & n_2(2n_2-1) \\ & \ldots 0 & 2g & (2n_2+1)(n_2+1)g+2+u \end{vmatrix}=0 \tag{2.12}$$

where the appropriate values of u follow for $n_1=0, 1, 2$ from (2.6), (2.8) and (2.9), respectively. Equation (2.12) represents an algebraic equation in g. In order to have pairs of solutions of the type (2.1), (2.2) this equation must have at least one positive g root.

3. Results

For $n_1=0$ and $1\leqslant n_2\leqslant 5$ the algorithm described above reproduces the reported results of Gallas (1988). In order to have a better statistics on the data we have numerically

Table 1. The positive g roots of (2.12) for the case $n_1 = 1$ together with their differences Δg. The root g_1 corresponds with the value of u (or λ) (2.8) with the minus sign; g_2 is related to the plus sign.

n_2	g_1	Δg_1	g_2	Δg_2
0	—		—	
1	—		—	
2	0.312 447		0.826 435	
		0.295 718		0.803 418
3	0.608 165		1.629 853	
		0.286 615		0.799 982
4	0.894 780		2.409 835	
		0.286 087		0.775 901
5	1.180 867		3.185 736	
		0.296 474		0.774 511
6	1.467 341		3.960 247	
		0.286 904		0.773 882
7	1.754 245		4.734 129	
		0.287 253		0.773 544
8	2.041 498		5.507 673	
		0.287 520		0.773 345
9	2.329 018		6.281 018	
		0.287 728		0.773 216
10	2.616 746		7.054 234	
		0.287 886		0.773 128
11	2.904 632		7.827 362	
		0.288 009		0.773 067
12	3.192 641		8.600 429	

Table 2. The positive g roots of (2.12) for the case $n_1 = 2$ together with their differences Δg. The roots g_1, g_2, g_3 are related to the three distinct negative u (or λ) roots of (2.9).

n_2	g_1	Δg_1	g_2	Δg_2	g_3	Δg_3
0	—		—		—	
1	—		—		—	
2	—		—		—	
3	0.192 650		0.377 376		0.976 190	
		0.164 566		0.299 738		0.821 791
4	0.357 216		0.677 114		1.797 981	
		0.168 030		0.286 911		0.784 505
5	0.525 246		0.964 025		2.582 486	
		0.172 594		0.285 325		0.777 724
6	0.697 840		1.249 350		3.360 210	
		0.174 942		0.285 464		0.775 437
7	0.872 782		1.534 814		4.135 647	
		0.176 144		0.285 908		0.774 420
8	1.048 926		1.820 720		4.910 067	
		0.176 807		0.286 348		0.773 890
9	1.225 733		2.107 070		5.683 957	
		0.177 205		0.286 722		0.773 580
10	1.402 938		2.393 792		6.457 537	

derived the positive values of g for which (2.12) becomes zero for a range of values of n_2 between 0 and 13. For this range of n_2 values only one positive g root per n_2 value is present, with the exception of $n_2=0$, where only the trivial solution $g=0$ is found. These g values are: 0.666 666 ($n_2=1$), 1.457 427 (2), 2.234 857 (3), 3.009 794 (4), 3.783 828 (5), 4.557 433 (6), 5.330 802 (7), 6.104 026 (8), 6.877 155 (9), 7.650 219 (10), 8.423 235 (11), 9.196 216 (12), 9.969 171 (13). As established by Gallas and confirmed here, these g values are spaced by an almost constant difference Δg: 0.790 761, 0.777 430, 0.774 937, 0.774 034, 0.773 605, 0.773 369, 0.773 224, 0.773 129, 0.773 064, 0.773 016, 0.772 981, 0.772 955.

It is quite easy to prove that for all n_2 values larger than zero at least one positive real g value exists when $n_1=0$. Substituting the expression (2.6) for u into (2.12) one can verify that the resulting polynomial equation has:

(i) one trivial $g=0$ root, which can be divided out;

(ii) after the elimination of this zero root a constant term given by $2^{n_2+1}(2n_2-1)n_2!$;

(ii) a coefficient of the highest-degree term denoted by $-(n_2+2)!(2n_2-1)!!$.

This means that the product of all non-zero roots of (2.12) is simply given by the expression

$$\frac{(-1)^{n_2+1}2^{n_2+1}n_2!}{(n_2+2)!(2n_2-3)!!}. \tag{3.1}$$

This means that for n_2 odd, one always obtains at least one positive real g root. For n_2 even the product of all roots (complex and real) is, by (3.1), negative; this means that at least one negative real g root and then also one positive real g root is present. Thus in all cases this simple reasoning shows that at least one positive real root can be found for (2.12). The numerical verification confirms the uniqueness of this positive root for $0<n_2\leq 13$.

For $n_1=1$ or 2, two or three distinct values of u (see (2.8) and (2.9)) are respectively available for each positive g value. This means that one can hope that for each of these u values (2.12) can be fulfilled for at least one positive g value. The positive roots of (2.12) have been determined numerically for $n_1=1, 2$ and for several values of n_2. These results, together with the Δg differences, are reported in table 1 and table 2, respectively. For the n_2 values considered, we find that for each u value, which is a solution of (2.7) ($n_1=1$) or (2.9) ($n_1=2$), there exists a positive g value for which pairs of solutions of the type (2.1), (2.2) of (1.1) can be constructed provided that $n_2>n_1$. The fact that the u values which occur are irrational expressions with respect to g makes it impossible to prove this statement in general with techniques analogous to those used for the $n_1=0$ case. We can also observe in table 1 and table 2 that the g values obtained are approximately spaced by a constant difference and that the Δg values found for the $n_1=0$ case approximately occur again for other n_1 values. The second Δg values present in the $n_1=1$ case (table 1) reappear in the $n_1=2$ case (table 2). It does not seem that the Δg converge to a constant as conjectured by Gallas, since, as can be observed in both tables, some of the tabulated Δg values pass through a minimum and then increase further on with increasing n_2.

4. Conclusions

The eigenproblem (1.1) admits solution pairs consisting of an odd- and an even-parity eigenfunction, both expressed analytically as the product of $(1+gx^2)\exp(-x^2/2)$ and a single polynomial as indicated in (2.1) and (2.2). From the cases considered, i.e. odd-parity eigenfunctions with polynomial parts of degree one ($n_1 = 0$), three ($n_1 = 1$) and five ($n_1 = 2$), we can conjecture that the polynomial part of the even-parity eigensolutions must be of degree $n_1 + 1$ or higher. For a chosen and fixed value of n_1, there exist $n_1 + 1$ relations between λ and g, such that $\lambda < 0$, $g > 0$ and $\lambda = \lambda(g)$. For each acceptable λ value, the cases under consideration show the possible existence of an infinite number of solution pairs. For $n_1 = 0$ we have proved that this is the case. For $n_1 > 0$ it should be very interesting to prove or disprove this conjecture and to give an explanation for the behaviour of the observed g spacing.

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